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CALCULATION OF TURBULENT BOUNDARY LAYERS OVER FLAT PLATES WITH DIFFERENT PHENOMENOLOGICAL THEORIES OF TURBULENCE AND VARIABLE TURBULENT PRANDTL NUMBER

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Abstract—Three different hypotheses representing turbulent transport are investigated. The three models are the Van Driest model of turbulence, a modified Nee–Kovasznay hypothesis and a combination of the kinetic energy of turbulence and the mixing length hypotheses. A model for the variable turbulent Prandtl number is developed. Numerical solutions are obtained using a modified Spalding–Patankar finite difference method. Several empirical constants are evaluated and predictions are then tested against experimental data and show very good agreement. The importance of a variable turbulent Prandtl number is demonstrated.

NOMENCLATURE

A, constant, equation (6);
A^*, Van Driest constant, equation (5);
a, constant, equation (7);
B, constant, equation (6);
C_D, dissipation constant, equation (8);
C_f, skin friction coefficient $= 2\tau_w/\rho U_e^2$;
C_s, turbulent diffusion constant, equation (9);
D, dissipation term, equation (7);
K, Von Kármán's constant;
l, Prandtl's mixing length;
$\dot{m}''$, wall mass flux;
n, total viscosity = eddy viscosity + kinematic viscosity;
Pr, Prandtl number;
q, kinetic energy of turbulence $= \frac{1}{2}\overline{u_i u_i}$;
$\dot{q}''$, wall heat flux;
Re, Reynolds number;
St, Stanton number $= \dot{q}''/\rho C_p U_e(T_w - T_e)$;
T, temperature;
T', fluctuation in temperature;
U, V, velocity components;
U^+, dimensionless velocity $= U/V^+$;
u, fluctuating velocity component;
V^+, friction velocity $= \sqrt{\tau_w/\rho}$;
x, coordinate in streamwise direction;
y, coordinate perpendicular to wall;
y^+, dimensionless distance $= yV^+/\nu$;
α, thermal diffusivity;
ε_H, turbulent eddy diffusivity for heat;
ε_M, turbulent eddy diffusivity for momentum;
ϕ, generalized dependent variable, equation (A.3);
λ constant, equation (5);
μ, dynamic viscosity;
ν, kinematic viscosity $= \mu/\rho$;
ω, cross stream coordinate, equation (A.2);
ρ, density;
τ, shear stress;
ψ, stream function;
θ, momentum thickness.

Notation

$(\ )_e$, free stream value;
$(\ )_E$, outer edge of boundary layer;
$(\ )_{\text{eff}}$, effective value;
$(\ )_i$, ith fluctuating velocity component;
$(\ )_I$, inner edge of boundary layer (wall);

$(\ )_j$, jth fluctuating velocity component;
$(\ )_t$, turbulent value;
$(\ )_w$, value at wall;
$(\ )_x$, value at the location x along the plate.

INTRODUCTION

IN RECENT years phenomenological theories of turbulent shear flows have been the interest of many authors [1–5]. The purpose of the theories is to yield a universal predictive capability by including more of the physics of turbulent motion and less empirical formulae. Further work is needed to supplement these original hypotheses with either additional postulated relationships or further evaluation of the various universal numerical constants.

Many investigators have made use of the empirical mixing length formulae and an excellent summary is given by Blom [6]. A fundamental objection to the use of empirical mixing length formulae, even in their most developed form (Spalding [7], Reichardt [8] and Van Driest [9]), is that they are valid only when local equilibrium exists between generation and dissipation of turbulent energy. A further objection is the fact that the Reynolds stress terms are only related to the local mean velocity field, while the effect of the past history of the boundary layer is ignored. These objections led to the Kolmogorov [10] and Prandtl [11] turbulent energy hypothesis relating the Reynolds stresses to the turbulent kinetic energy which is governed by a rate equation. In this way the local state of the turbulence is related to the other turbulent properties such as the length scale and the kinetic energy of the fluctuations.

Improvements were made to the mixing length approaches by Patankar and Spalding [4, 12] wherein a modified Van Driest [9] eddy viscosity model was proposed. This is a model in which the local damping is affected by the local shear rather than the wall shear. The problem of history, however, still remained.

Townsend [13] proposed a turbulent constitutive equation that relates the fluctuation correlations to the turbulence energy and the mean flow field which allows boundary layer history to be brought into view. Wolfshtein [5] used Townsend's equation in a one-dimensional form to obtain velocity and temperature profiles for Couette flow. Kearney *et al.* [14] solved Townsend's equation for turbulent kinetic energy in the outer region. Their primary interest was in the effect of free stream turbulence on heat transfer and they presented Stanton number results only.

Harlow and Nakayama [3] have also developed a turbulent kinetic energy equation as well as an equation which describes the local turbulent energy dissipation. Ziemniak [15] used Harlow and Nakayama's set of equations to investigate two-dimensional channel flow. No attempt has been made to use their equations for a boundary layer flow.

A different approach was taken by Nee and Kovasznay [1, 2] wherein they postulate a rate equation to govern the effective viscosity. The effects of advection, diffusion, generation and decay are each represented by an appropriate term leaving two empirical constants to be determined by experiment.

To apply the various turbulent transport theories to heat or mass transfer, one must specify a turbulent Prandtl or Schmidt number. Most workers have solved the energy equation by assuming a constant value of Pr_t of about 0·8 or they assume Reynolds analogy would hold. Patankar and Spalding [4] have suggested that Pr_t remains uniform at a value near 0·9. Powell and Strong [16] have also assumed constant turbulent Prandtl number of 0·9. A review of the published experimental values of Pr_t by Blom [6] and Simpson *et al.* [17] demonstrates that a constant value of Pr_t is incorrect. Experimental results [6, 17] have clearly shown that Pr_t is a function of distance from the wall.

The problem to be investigated in this work is a stationary, two-dimensional, incompressible flow over a flat plate with negligible viscous dissipation. The overall temperature difference $(T_w - T_e)$ will be assumed small such that the fluid properties may be taken constant at some

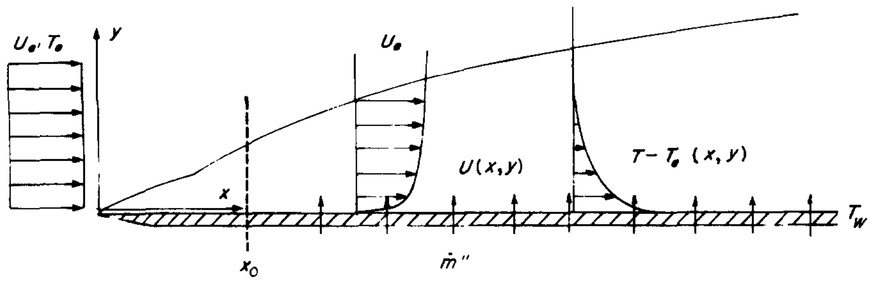

FIG. 1. Sketch of problem investigated.

average temperature. A sketch of the flow situation is presented in Fig. 1. The present work will compare the predictions of the following models for the eddy viscosity:

(1) The Van Driest mixing length model utilized throughout the entire boundary layer.

(2) A rate equation governing the distribution of the total viscosity, $n = \nu + \varepsilon_M$, solved in the outer region with the inner region being described by an eddy viscosity model.

(3) An equation describing the turbulent kinetic energy solved in an outer region with the eddy viscosity being extracted through a Prandtl–Wieghardt type relation and the near wall region described by an eddy viscosity model.

Further, the commonly used model for turbulent heat transfer will be modified to account for the variation of turbulent Prandtl number across the layer. This will yield a correlation that fits the available data of Pr_t within the limits of the uncertainty envelope [17]. The effect of variable turbulent Prandtl number on heat transfer predictions will also be presented. It should be noted that the above mentioned developments also apply for turbulent mass transfer if one introduces turbulent Schmidt number instead of turbulent Prandtl number.

GOVERNING EQUATIONS

Incompressible, two-dimensional, steady, constant-property turbulent boundary layer flow on a flat plate is governed by the following equations

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{1}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \varepsilon_M)\frac{\partial U}{\partial y}\right] \tag{2}$$

$$U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \varepsilon_M)\, Pr_{\text{eff}}^{-1}\frac{\partial T}{\partial y}\right] \tag{3}$$

where the effective Prandtl number is defined by

$$Pr_{\text{eff}} = \frac{\nu + \varepsilon_M}{\alpha + \varepsilon_H} = \frac{1 + (\varepsilon_M/\nu)}{Pr^{-1} + Pr_t^{-1}(\varepsilon_M/\nu)}, \tag{4}$$

where $Pr_t = \varepsilon_M/\varepsilon_H$ is the turbulent Prandtl number. In the fully turbulent part of the boundary layer $\varepsilon_M \gg \nu$, that is $Pr_{\text{eff}} = Pr_t$ while in the viscous sublayer $\varepsilon_M \ll \nu$ and $Pr_{\text{eff}} = Pr$.

Models of turbulence

Van Driest, Patankar and Spalding model. This model involves a near-wall-region formulation and another formulation for the outer region of the boundary layer. In the outer part, the mixing length, l, is taken as uniform and proportional to the thickness of the layer, while near the wall, l is proportional to distance from the wall. Thus,

$$\varepsilon_M = k^2y^2\left|\frac{\partial U}{\partial y}\right|[1 - \exp(-y\sqrt{(\rho\tau)}/\mu A^*)]^2 \quad \text{for } 0 < yk < \lambda y_l, \tag{5a}$$

$$\varepsilon_M = \lambda^2 y_l^2\left|\frac{\partial U}{\partial y}\right| \quad \text{for } \lambda y_l < yk \tag{5b}$$

where λ, k and A^* are the undetermined constants.

Nee–Kovasznay model. A rate equation for total viscosity, $n = \nu + \varepsilon_M$, has been developed for the outer region. Combining this equation with the momentum equation forms a closed system that introduces two non-dimensional universal constants to be determined by matching calculated solutions to experiments.

It is assumed that the total viscosity, n, obeys a rate equation of the form

$$U\frac{\partial n}{\partial x} + V\frac{\partial n}{\partial y} = \frac{\partial}{\partial y}\left(n\frac{\partial n}{\partial y}\right) + A(n-\nu)\left|\frac{\partial U}{\partial y}\right| - \frac{B}{y^2}n(n-\nu) \tag{6}$$

where A and B are the two undetermined constants.

Equation (6) accounts for convection, diffusion, generation and decay of turbulence. Furthermore, it is assumed that turbulent diffusion occurs with n being its own exchange coefficient.

Nee and Kovasznay [2] suggested that for the outer region $y^+ \geqslant y_m^+$, the velocity field U and the total viscosity field n are solved directly from the governing equations (1), (2) and (6). For the inner region, $y^+ \leqslant y_m^+$, the velocity field is described by the linear and logarithmic laws of the wall, In addition, a linear growth of the boundary layer thickness was assumed. In the present work, the total viscosity field n is solved from (5a) and (5b) in the inner region, $y^+ \leqslant y_m^+$, and (6) in the outer region, $y^+ \geqslant y_m^+$. The dimensionless distance y_m^+ is determined by matching the calculated solution to experimental data.

Kinetic energy of turbulence. It was mentioned earlier that the objection to the mixing length hypothesis has led to the Kolmogorov–Prandtl [10, 11] turbulence energy hypothesis, where the local state of turbulence is assumed to depend on a length scale and on the kinetic energy of the velocity fluctuations ($q = \frac{1}{2}\overline{u_i u_i}$). In an early phase of the present work, Harlow and Nakayama's turbulence transport equations [3] were the main task. Two transport equations that govern the turbulent kinetic energy and dissipation of turbulent kinetic energy were to be solved. However, a system of large number of equations containing a large number of constants (assuming they are universal) and postulated relationships was not easy to solve. In this work, a postulated relationship for the dissipation term is, therefore, introduced instead of solving an additional rate equation for the dissipation of the kinetic energy of the fluctuations.

The model chosen here is a combination of the kinetic energy hypothesis in the outer region and the Van Driest model of turbulence in the inner region. In other words, for $0 < y^+ \leqslant y_m^+$, ε_M is given by equations (5a) and (5b) and, for $y^+ > y_m^+$, ε_M is given by

$$U\frac{\partial q}{\partial x} + V\frac{\partial q}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + a\varepsilon_M)\frac{\partial q}{\partial y}\right] + \varepsilon_M \times \left(\frac{\partial U}{\partial y}\right)^2 - D \tag{7}$$

together with an expression relating the turbulent kinetic energy and dissipation,

$$D = C_D q^{\frac{3}{2}}/y \tag{8}$$

and an expression relating the turbulent kinetic energy and the eddy viscosity,

$$q = \left(\frac{1}{C_s}\frac{\varepsilon_M}{y}\right)^2 \tag{9}$$

where a, C_D and C_s are universal constants to be evaluated by comparison with experimental results. y_m^+ is taken to be 30 for equation (7) to be in the fully turbulent part of the boundary layer ($y^+ \geqslant 30$). It is of importance to mention that in the turbulent core of the layer the length scales for turbulent diffusion and dissipation are identical and equal to the distance from the wall.

Boundary conditions

The conservation of mass and momentum, equations (1) and (2), require specification of the velocity components at the wall and at the free-stream. That is,

$$y = 0: U = 0, V = \dot{m}''/\rho$$
$$y \to \infty: U = U_e. \tag{10}$$

The temperature form of the thermal energy equation requires specification of the temperature at the wall and in the freestream

$$y = 0: T = T_w$$
$$y \to \infty: T = T_e. \tag{11}$$

For the thermal energy equation the heat flux at the wall can be specified as an alternative for the Dirichlet boundary condition.

The total viscosity, n, and the turbulent kinetic energy, q, boundary conditions are

$$y = 0: n = \nu, q = 0$$
$$y \to \infty: n = \nu, q = 0. \tag{12}$$

Alternatively, the freestream turbulence may be specified at the boundary layer edge.

METHOD OF SOLUTION

The numerical procedure employed is a modification of the Spalding–Patankar procedure [4, 12] developed by Denny and Mills [18]. The method solves the finite-difference-versions of the nonlinear parabolic partial differential equations in a step-wise manner in the downstream direction.

In the original work of Patankar and Spalding [4, 12] the near wall region was given a special treatment. A Couette flow analysis was carried out and the resulting ordinary differential equations yielded the "slip value" relations. The finite difference scheme was matched to these slip values at some point in the layer.

In the present work the slip value scheme has been discarded. The near wall region detail is maintained by the finite difference scheme once an appropriate initial profile is given at the starting point. In addition, the use of a variable cross-stream step size yields accurate results near the wall. Furthermore, Denny and Landis [19] suggested a modified transformation, the ω^2 transformation, and compared it with the standard ω-transformation. They have shown that, the ω^2-transformation yields more accurate results due to the well-behaved nature of the solution near the wall in the transformed plane. Also, more accurate computed wall gradients are attainable due to the reduced truncation error involved in the finite difference analogue of the differential equations. Further computational details are given in Appendix A.

TURBULENT PRANDTL NUMBER

Experimental results have shown that Pr_t is not a constant across the boundary layer but a function of the distance from the wall [6, 17]. Pr_t can be determined by measurements of velocity and temperature distributions in the boundary layer.

Blom [6] and Simpson *et al.* [17] presented a survey of experimental values of Pr_t. The survey shows that Pr_t values are widely scattered even for the same Pr. Kestin and Richardson [20] showed that mercury experiments in pipes indicated that $Pr_t > 1$ while gas experiments in pipes showed $Pr_t < 1$. It is not clear whether Pr_t is completely independent of the molecular Prandtl number.

Experimental values of Pr_t obtained by Simpson *et al.* [17] have shown that near the wall ($y^+ < 150$) the local value of $Pr_t > 1$. Near the wall Pr_t depends on Pr since the molecular viscosity and Prandtl number govern momentum and heat transport. Rotta [21] suggested that for the outer region ($y/\delta > 0{\cdot}1$) Pr_t can be expressed by

$$Pr_t = 0{\cdot}95 - 0{\cdot}45\,(y/\delta)^2. \tag{13}$$

Blom [6], however, presented a completely different trend for Pr_t. He showed that Pr_t increases with increasing y^+ for small values of y^+, is nearly constant for intermediate y^+ values, and decreases in the outer region.

For the calculation of heat transfer, the distribution of Pr_t in the inner region is of primary importance, while in the outer region the distribution of Pr_t is of secondary importance, for the largest resistance to heat transfer is concentrated near the wall. Blom [6] showed that in the inner region of the thermal boundary layer $Pr_t < 1$. Furthermore, a universal distribution for Pr_t in the near wall region exists and has the form,

$$Pr_t = \frac{e^{KU^+} - 1 - KU^+ - (KU^+)^2/2! - (KU^+)^3/3! - (KU^+)^4/4!}{e^{KU^+} - 1 - KU^+ - (KU^+)^2/2! - (KU^+)^3/3!} \quad (14)$$

with $K = 0{\cdot}4$.

Model for turbulent Prandtl number

The turbulent heat flux is commonly expressed as

$$\overline{u_i T'} = -\text{constant} \times \varepsilon_M \frac{\partial \overline{T}}{\partial x_i} \quad (15)$$

where the constant is Pr_t^{-1}. However, to account for the non-constancy of Pr_t across the layer one needs to modify the model to one that satisfies the following requirements:

(1) Accounts for the unequal loss of momentum and thermal energy from an eddy during its motion from an initial point to a nearby one.

(2) Yields an expression for Pr_t that depends on the laminar Pr in the inner region (low intensity region). For the largest resistance to heat transfer, which is governed by molecular Prandtl number, is concentrated near the wall.

(3) In the outer region (high intensity region) the turbulent heat flux is expressed by an expression similar to that given by (15), i.e. independent of Pr.

(4) Predicts Pr_t values that fit the available data.

To accomplish this, consider the diffusion of thermal energy from an eddy during its travel between two points before it breaks up and mixes with the surrounding fluid. One can reach the following expression

$$\overline{u_i T'} = -C_1 \frac{\varepsilon_M^2}{\alpha}\left[1 - \exp\left(-C_2 \frac{\alpha}{\varepsilon_M}\right)\right]\frac{\partial \overline{T}}{\partial x_i}. \quad (16)$$

One can derive an expression for $\overline{u_i u_j}$ similar to equation (16). Accordingly, the turbulent Prandtl number is given by

$$Pr_t = \frac{C_3}{C_1 Pr} \frac{\left[1 - \exp\left(-\dfrac{C_4}{(\varepsilon_M/\nu)}\right)\right]}{\left[1 - \exp\left(-\dfrac{C_2}{Pr(\varepsilon_M/\nu)}\right)\right]} \quad (17)$$

where C_1, C_2, C_3 and C_4 are constants. It can be seen that for $(\varepsilon_M/\nu) \to 0$, $Pr_t = C_3/C_1 Pr$ and for $(\varepsilon_M/\nu) \to \infty$, $Pr_t = C_3 C_4/C_1 C_2$, independent of Pr. The above also applies to turbulent Schmidt number.

RESULTS

Results are presented for turbulent boundary layers of air on a flat plate. Heat transfer results were obtained with a wall to free stream temperature ratio, T_w/T_e, of 1·04. A value of 0·9 for the turbulent Prandtl number is utilized for the constant Prandtl number results.

Evaluation of the empirical constants

In the presentation of the different models, it was shown that each postulation involves some empirical constants (assuming they are universal). These constants are evaluated by matching the computed solutions to reliable experimental data. Comparisons with the data of Wieghardt [22] were carried out. This data has been accepted as being reliable by the Stanford Conference on turbulent boundary layers [22]. Accordingly, the following were obtained:

(1) The constants λ, k and A^* in the Spalding–Patankar model (S–P) were taken to be 0·09, 0·435 and 26 respectively. These values, suggested by the original authors [4, 12], were found to be the best fit to data.

(2) For the Nee–Kovasznay model (N–K) A and B were taken to be 0·1 and 1·0 respectively. The values of y_m^+ was obtained by numerical experiments. A good fit was obtained with $y_m^+ = 200$. The effect of y_m^+ on the skin friction is shown in Fig. 2.

(3) The constants a, C_D and C_s in the turbulent energy model (K–E) were found to be 0·65, 0·42 and 0·1455 respectively. These constants were evaluated by numerical experiments to

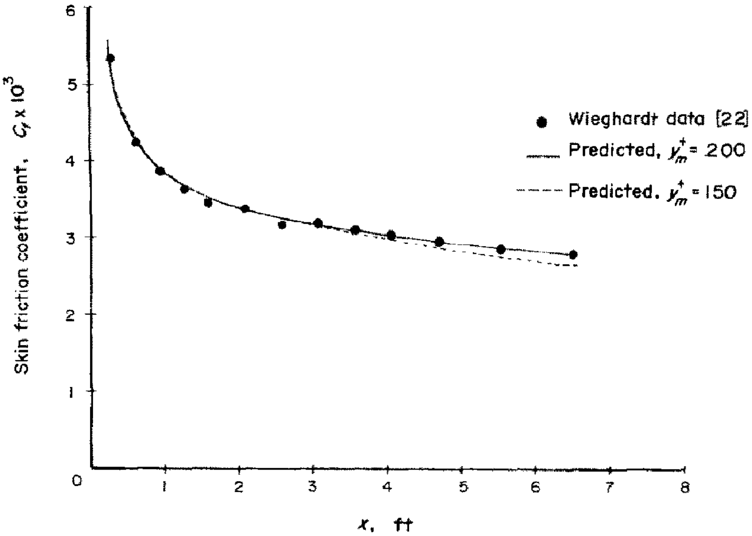

FIG. 2. Effect of y_m^+ on skin friction for (N–K) model.

yield the best fit possible to the experimental displacement thickness, skin friction, and velocity profiles. y_m^+ was chosen to be 30 for equation (7) to be within the turbulent core of the boundary layer. It should be mentioned that values of a from 0·4 to 1·2 can be found in the literature [5,15].

Comparison with experimental data

Skin friction results are presented in Fig. 3 for the three models (S–P), (N–K) and (K–E).

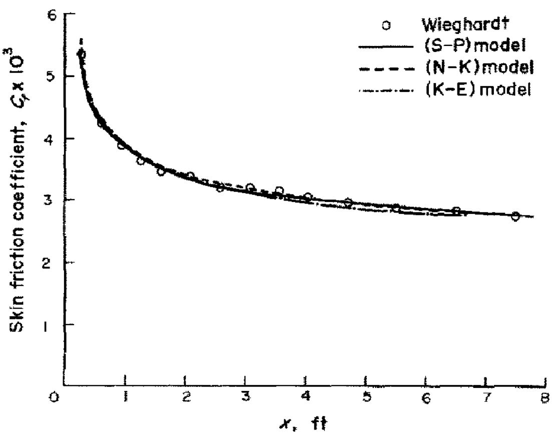

FIG. 3. Comparison of computed skin friction with Wieghardt's data [22].

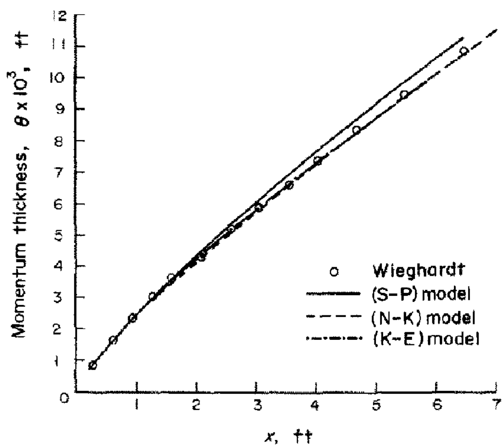

FIG. 4. Comparison of computed momentum thickness with Wieghardt's data [22].

Excellent agreement with the Wieghardt data [22] for all models is shown.

Momentum thickness predictions are shown in Fig. 4 for the different models. Excellent agreement with Wieghardt data [22] are obtained for models (N–K) and (K–E). For model (S–P) the predicted momentum thickness starts to deviate from the data at higher Re_x. However the prediction is still within 2–3 per cent of the data at high Re_x.

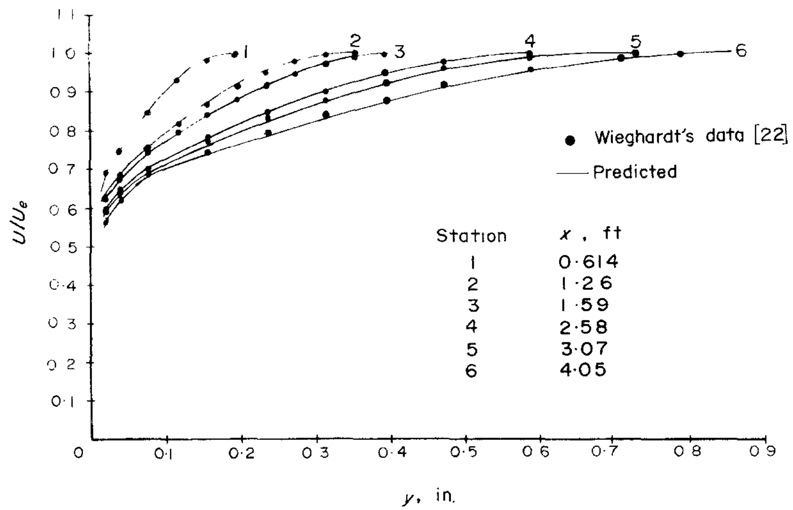

FIG. 5. Comparison of calculated and experimental velocity profiles, (S–P) model.

Mean velocity profiles are presented in Figs. 5–7 at different locations along the plate. Excellent agreement with Wieghardt data [22] is given by the (S–P) model. Good agreement is given by the (N–K) model. The profile tends to coincide with experimental data at higher Re_x as shown in Fig. 6. Velocity profiles predicted by the (K–E) model are found to be flatter than the data for approximately $0{\cdot}1 < y/\delta < 0{\cdot}5$ as shown in Fig. 7.

Eddy viscosity profiles predicted by the different models are shown in Figs. 8–10 at different stations along the plate. Values of ε_M/ν computed by both (S–P) and (N–K) models are almost the same. The (K–E) model gives higher predictions for ε_M/ν than the others. Figures 8–10 show that a universal profile for ε_M/ν vs y^+ does exist for $y^+ < 100$.

Dimensionless velocity results are presented in Figs. 11–13 for the different models. The

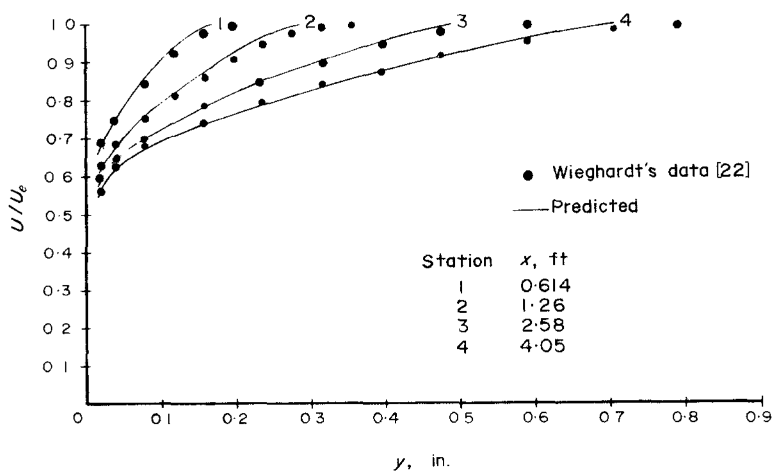

FIG. 6. Comparison of calculated and experimental velocity profiles, (N–K) model.

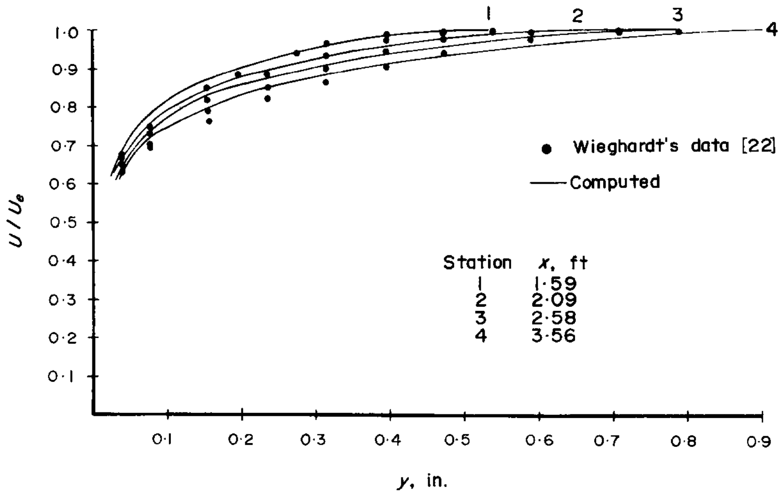

FIG. 7. Comparison of calculated and experimental velocity profiles, (K–E) model.

(S–P) and (N–K) models give good agreement with Wieghardt's data [22]. Predictions given by the (K–E) model are higher than the data for intermediate values of y^+.

Heat transfer predictions for constant turbulent Prandtl number are presented in Fig. 14 for a temperature ratio of 1·04 and a turbulent Prandtl number of 0·9. Results are compared with the data of Moffat and Kays [23]. The (S–P) model prediction is within 4 per cent of the experimental data. The predictions of both the (N–K) and the (K–E) models are in very good agreement with the data.

Turbulent Prandtl number results

Predicted values of Pr_t are presented in Figs.

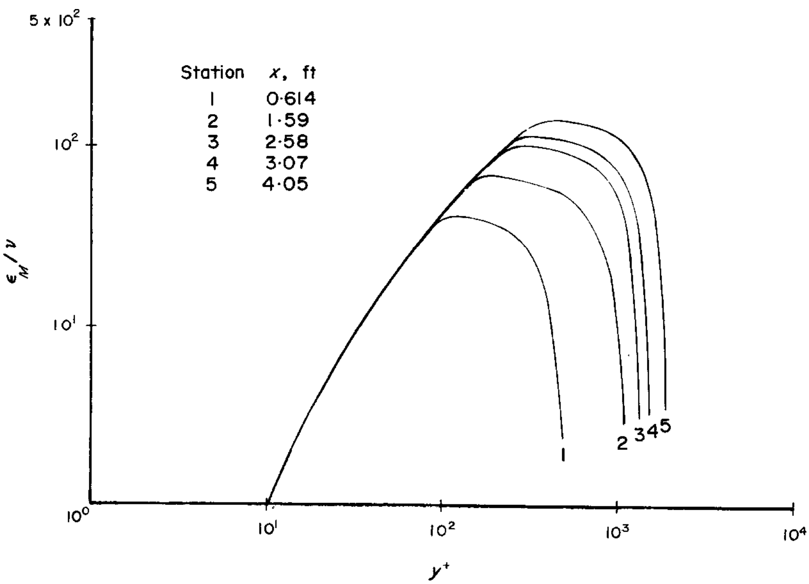

FIG. 8. Computed eddy viscosity distribution, (S–P) model.

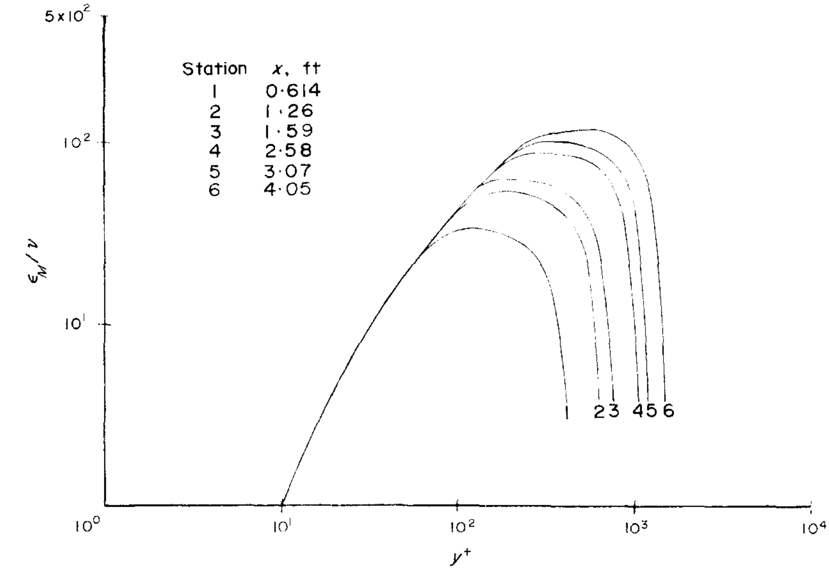

FIG. 9. Computed eddy viscosity distribution, (N–K) model.

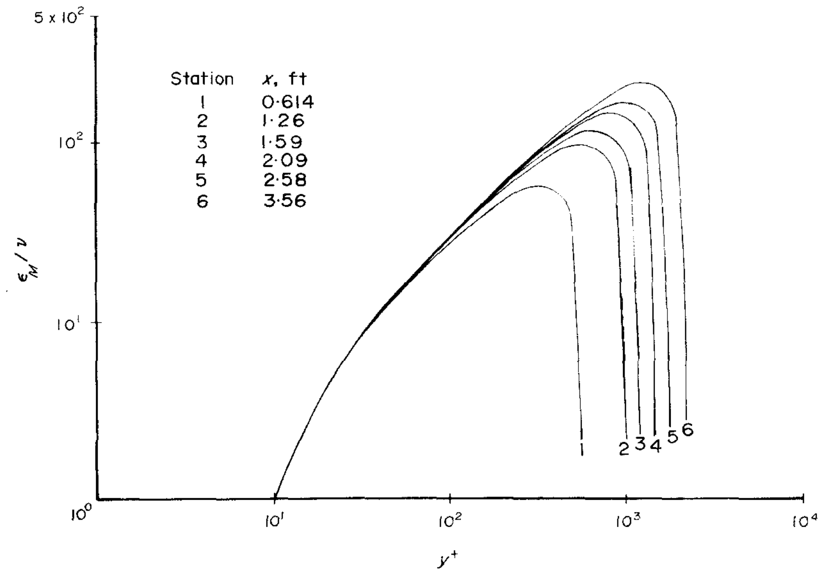

FIG. 10. Computed eddy viscosity distribution, (K–E) model.

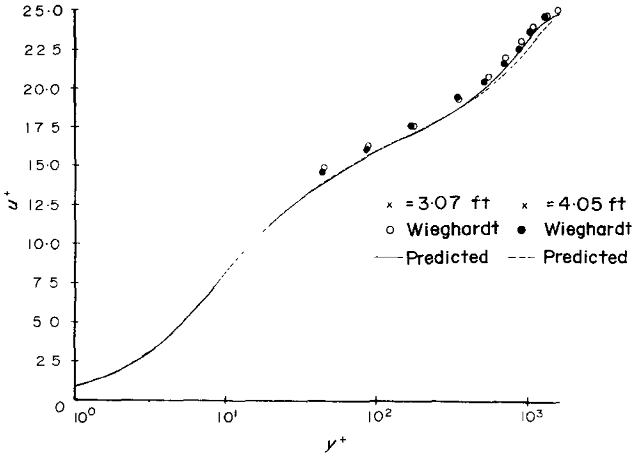

FIG. 11. Comparison of calculated dimensionless velocity and Weighardt's data [22], (S–P) model.

15 and 16 and compared with the data of Blom [6] and Simpson *et al.* [17]. The constants C_1, C_2, C_3 and C_4 were chosen to correlate with the data. Numerical experiments show that if $C_1 = 0{\cdot}21$, $C_2 = 5{\cdot}25$, $C_3 = 0{\cdot}20$ and $C_4 = 5$, Pr_t values will fall within the uncertainty envelope of the experimental results (Fig. 16). Predictions are also compared with equations (13) and (14) as depicted in Fig. 15.

Heat transfer prediction with variable turbulent Prandtl number

Figure 17 shows the calculated curves of Stanton number against Reynolds number for both constant and variable turbulent Prandtl number. The prediction is for a temperature ratio, T_w/T_e, of 1·04 and the Van Driest model of turbulence. A value of 0·9 for the constant turbulent Prandtl number is recommended by

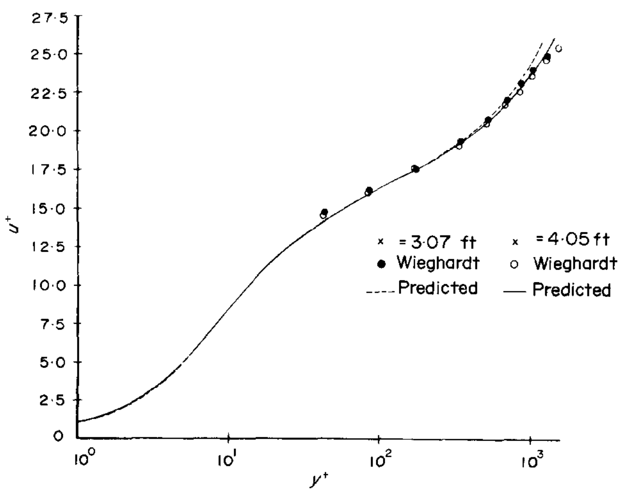

FIG. 12. Comparison of calculated dimensionless velocity and Weighardt's data [22], (N–K) model.

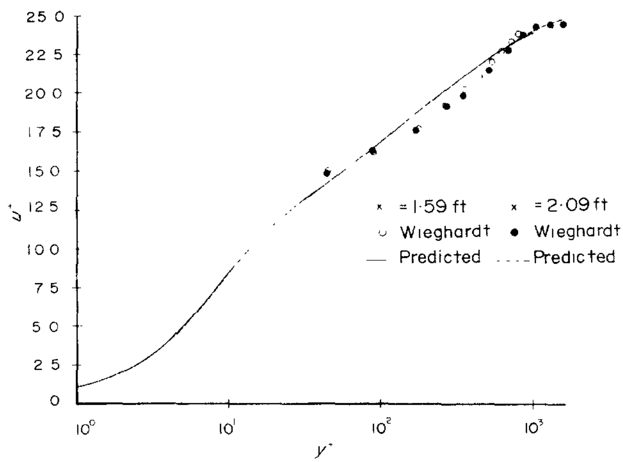

FIG. 13. Comparison of calculated dimensionless velocity and Weighardt's data [22], (K–E) model.

Spalding and Patankar [4] is used. The variable turbulent Prandtl number is computed by equation (17) with the numerical constants presented above. Figure 17 indicates that an improved prediction for the heat transfer to the wall is obtained if one considers the true variation in Pr_t across the layer rather than the commonly used approach of adopting a constant value of Pr_t.

DISCUSSION OF RESULTS

Three phenomenological theories of turbulence have been investigated and compared with experiment in their original or somewhat modified form. An expression for variable turbulent Prandtl number is presented.

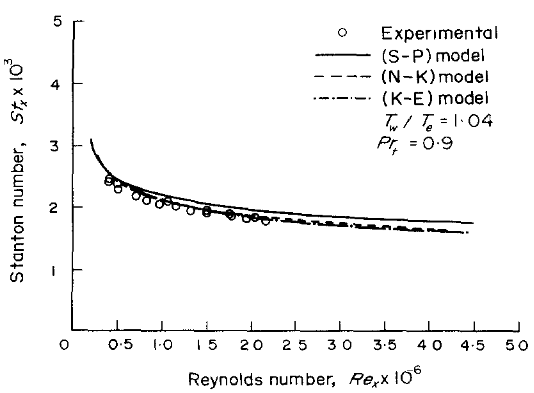

FIG. 14. Comparison of computed Stanton number with Moffat and Kay's data [23].

It was found that Van Driest's modified mixing length hypothesis [4] yielded excellent agreement with Wieghardt's data for skin friction coefficient and the velocity profiles. The predicted momentum thickness is higher than the experimental values at high Reynolds number. The model does, however, suffer from the following drawbacks:

(1) It relates the state of turbulence of the fluid to the local length scale and the local mean velocity field. Hence ignoring the past history of the boundary layer.

(2) Zero eddy viscosity in regions of zero mean velocity gradient.

(3) It is only valid in the turbulent part of the boundary layer where the generation and dissipation of turbulent kinetic energy are in equilibrium.

The Nee–Kovasznay theory [1] takes into account the relevant mechanism of turbulent motion but only with the minimum complications. The equation accounts for the effects of convection, diffusion, generation, and decay of eddy viscosity based on simple analogy with the kinetic energy of turbulence. Nee and Kovasznay [1, 2] solved equation (6) in an outer region

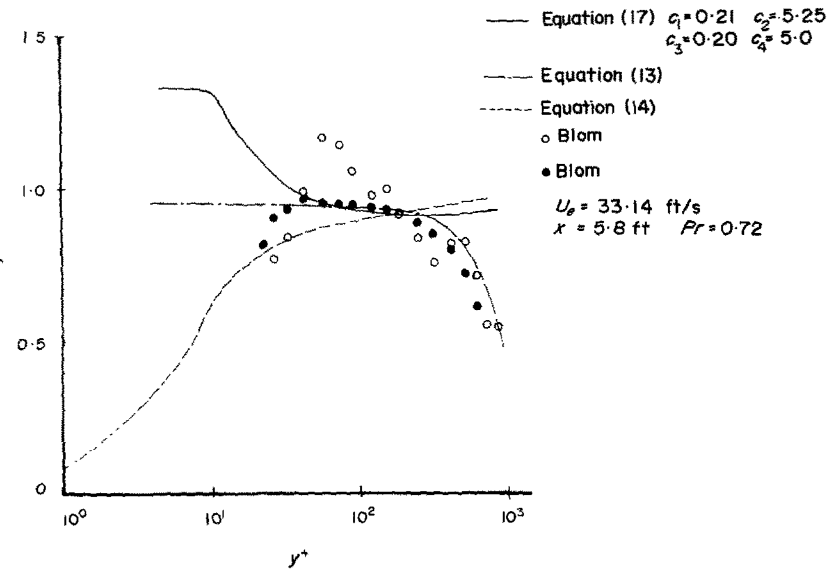

FIG. 15. Comparison of predicted Pr_t and data of Blom [6].

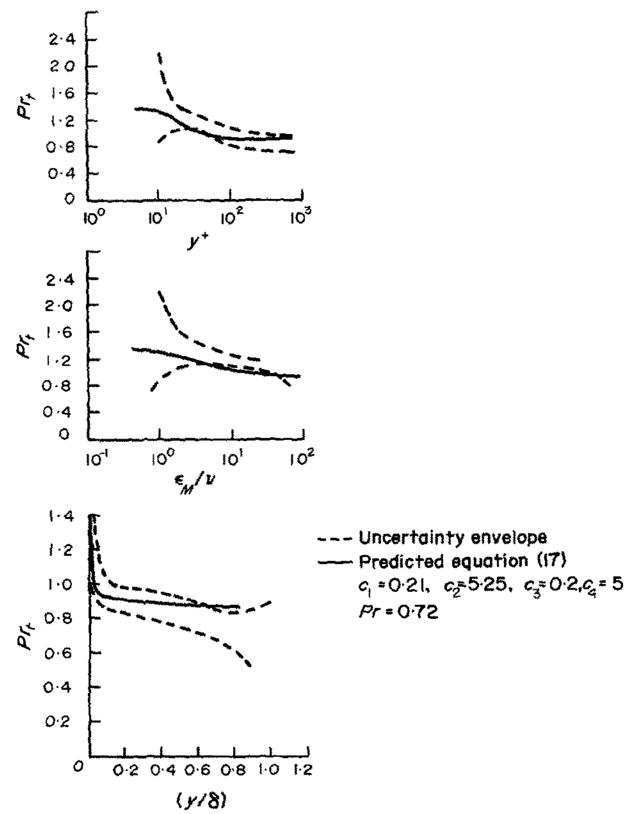

FIG. 16. Comparison of predicted turbulent Prandtl number and experimental data of Simpson *et al.* [17].

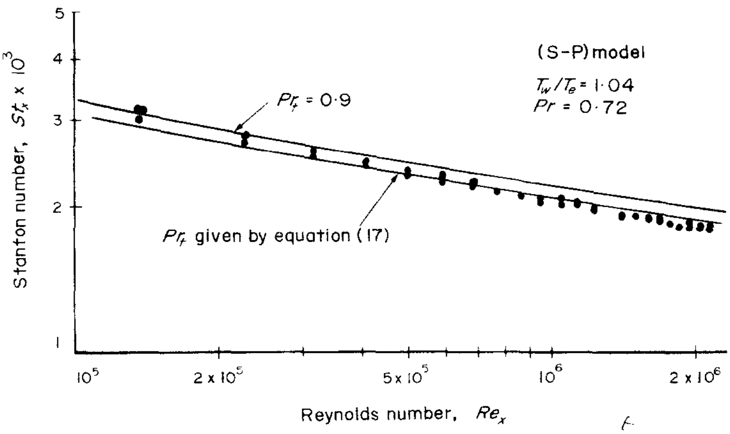

FIG. 17. Comparison of computed Stanton number with Moffat and Kay's data [23].

$(y^+ > y_m^+)$. In the inner region $(y^+ < y_m^+)$, the linear and logarithmic laws of the wall were assumed to describe the velocity profile. The linear law is based on a pure laminar sublayer adjacent to the wall. The concept of a laminar sublayer has been disproved as a result of measurements by Klebanoff [24] and Laufer [25]. The logarithmic law assumes that $\varepsilon_H \gg \nu$. Furthermore, both laws assume one dimensional analysis. To overcome these drawbacks, in the present work, the momentum equation was solved with an adopted turbulent model in the inner region $(y^+ < y_m^+)$. The original values of the universal constants A and B suggested by Nee and Kovasznay [1] ($A = 0{\cdot}133$ and $B = 0{\cdot}8$) were also tested; but the revised values ($A = 0{\cdot}1$ and $B = 1$) rendered improved results. Excellent agreement of predictions with experimental values of skin friction, momentum thickness, and dimensionless velocity profiles [22] was found. Results were obtained for $y_m^+ = 200$. This value was determined by numerical experiments for the best fit to data (Fig. 2).

Nee–Kovasznay's equation should be extended to the wall to make sure that there are no inconsistencies in the model. It should be mentioned here that in an early phase of the present work efforts were made to extend the theory to the wall. Unfortunately, instability in the numerical solution was encountered. Further work will be done along this line.

The kinetic energy hypothesis which relates the local state of turbulence to a length scale and to the kinetic energy of fluctuations avoids the localness from which the mixing length hypothesis suffers. In deriving the kinetic energy of turbulence equation, one has to relate the Reynolds stress to the kinetic energy of fluctuations and to the rate-of-strain tensor. Also several transport flux approximations have to be introduced. This results in a large system of numerical constants and postulated relationships. Although the kinetic energy hypothesis provides more insight into the mechanisms of turbulence, it is difficult to apply. A system of equations involving several numerical constants and assumed functional forms is not easy to handle. In the present work unsuccessful efforts were made to apply Harlow–Nakayama's [3] theory to two-dimensional turbulent boundary layers, by solving two additional conservation equations governing the kinetic energy and the scale function. For this reason, the scale equation was deleted and an empirical postulation for the dissipation term was introduced.

In principle, the kinetic energy hypothesis is valid for the viscous sublayer as well as in the fully turbulent region. However, in the present work the kinetic energy equation is solved only in an outer region. This was done mainly to overcome the instability of the numerical solution due to the very steep gradient of the kinetic energy in the wall region as well as to make use of the fact that in the outer region ($y^+ > 30$) the length scale for turbulent diffusion and dissipation are identical and equal to the distance from the wall, y. By so doing, two additional universal constants are eliminated which reduces the complexity of the problem.

In spite of the drawbacks from which the Van Driest model of turbulence suffers as a result of the crudeness of the assumed mixing process, results presented show good predictions of turbulent boundary layers over flat plates. The model is fairly simple to handle and gives good agreement with experimental data regarding the prediction of skin friction and heat transfer to the surface. Normally, for practical applications, one is interested in the wall fluxes rather than in predicting the profiles and not aiming at highly detailed predictions of the local turbulence field. The Nee–Kovasznay theory, although it involves a better description of the turbulent mechanism than that of the Van Driest model, is not easy to handle. One has to solve the rate equation for the total viscosity in an outer region and a different model in the inner region. A better description of the turbulent mechanism is given by the kinetic theory of turbulence hypothesis. Due to the additional postulations required and the empirical constants involved, this model is more difficult than the others. Results presented for this model are not as good as those of the others. This is mainly due to the fact that more numerical experiments are still needed for the fine adjustment of the numerical constants associated with the theory. In general the Nee–Kovasznay model, with an improved turbulent model for $y^+ < y_m^+$, renders the best results of all.

In spite of the fact that the turbulent Prandtl number is not a constant across the boundary layer, most authors [4, 5, 12, 16] have solved the thermal energy equation by adopting a constant value of Pr_t. Experiments [6, 17] have shown that Pr_t can be $\gtreqless 1$ and a constant value of Pr_t is incorrect. The model developed in this work predicts Pr_t values within the uncertainty limits of the experimental results. In addition, the expression for Pr_t has a much simpler form than the Jenkins model. Heat transfer Stanton number results shown indicate that improved results are obtained with variable Prandtl number.

Finally, one can conclude that unless Nee–Kovasznay and kinetic energy of turbulent hypothesis are extended to the wall, the Van Driest model should be used because the computations are simpler and much faster. Furthermore, a lot of work is still required to test the universality of the empirical constants appearing in these hypotheses for different flow situations. A better approach to the problem would be by solving a set of equations describing the transport of momentum, turbulent kinetic energy, and dissipation of turbulent kinetic energy [3]. Further extension to the theory can be done for anisotropic turbulence by solving a full Reynolds stress transport equation, thus eliminating the necessity of postulating a relationship between turbulent kinetic energy and eddy viscosity.

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APPENDIX

Conservation Equations in x ~ ω Coordinates

From the definition of the stream function ψ, we have:

$$\frac{d\psi_I}{dx} = -\dot{m}''_I = -\rho V_I, \frac{d\psi_E}{dx} = -\dot{m}''_E = -\rho V_E \qquad (A.1)$$

where subscripts I and E denote the inner and outer edge of the boundary layer respectively. Introducing

$$\omega^2 = \frac{\psi - \psi_I}{\psi_E - \psi_I} = \frac{\int_0^y \rho u \,dy}{\int_0^{y_1} \rho u \,dy} \qquad (A.2)$$

allows us to transform equations into a standard form suitable for solution by the finite difference method, that is,

$$\frac{\partial \phi}{\partial x} + (a + b\omega^2)\frac{\partial \psi}{\partial \omega} = \frac{\partial}{\partial \omega}\left(c\frac{\partial \phi}{\partial \omega}\right) + d. \qquad (A.3)$$

ϕ stands for any of the dependent variables, a, b and c are variable coefficients and d is the source term. The solution is performed by obtaining the finite-difference equivalent of equation (A.3) and solving by marching and successive substitution techniques.

CALCUL DE COUCHES LIMITES TURBULENTES SUR PLAQUE PLANE AVEC DIFFERENTES THEORIES PHENOMENOLOGIQUES DE TURBULENCE ET UN NOMBRE DE PRANDTL TURBULENT VARIABLE

Résumé—On a étudié trois hypothèses différentes de transport par turbulence. Les trois modèles sont celui de Van Driest, une hypothèse modifiée de Nee-Kovasznay et une combinaison de l'énergie cinétique de turbulence et des hypothèses de longueur de mélange. On a développé un modèle pour le nombre variable de Prandtl turbulent. Des solutions numériques sont obtenues par utilisation d'une méthode modifiée aux différences finies de Spalding–Patankar. Plusieurs constantes empiriques sont évaluées et la comparaison des estimations aux résultats expérimentaux montre un très bon accord. On prouve l'importance d'un nombre de Prandtl turbulent variable.

BERECHNUNG VON TURBULENTEN GRENZSCHICHTEN ÜBER FLACHEN PLATTEN MIT VERSCHIEDENEN PHÄNOMENOLOGISCHEN TURBULENZTHEORIEN UND VARIABLER TURBULENTER PRANDTL-ZAHL

Zusammenfassung—Drei verschiedene Hypothesen, die den Turbulenztransport beschreiben, wurden untersucht. Die drei Modelle sind das Turbulenzmodell von van Driest, eine modifizierte Hypothese von Nee–Kovasznay und eine Kombination von kinetischer Turbulenzenergie mit einer Mischlängenhypothese.

Es wurde ein Modell für die variable turbulente Prandtl-Zahl entwickelt. Man erhielt numerische Lösungen unter Verwendung einer modifizierten finiten Differenzenmethode nach Spalding–Patankar.

Einige empirische Konstanten wurden geschätzt. Die berechneten Werte wurden dann mit experimentellen Daten verglichen. Die Übereinstimmung ist gut. Die Bedeutung einer variablen turbulenten Prandtl-Zahl wurde gezeigt.

РАСЧЕТ ТУРБУЛЕНТНОГО ПОГРАНИЧНОГО СЛОЯ НА ПЛОСКОЙ ПЛАСТИНЕ НА ОСНОВЕ РАЗЛИЧНЫХ ФЕНОМЕНОЛГИЧЕСКИХ ТЕОРИЙ ТУРБУЛЕНТНОСТИ, УЧИТЫВАЮЩИХ ИЗМЕНЕНИЕ ТУРБУЛЕНТНОГО ЧИСЛА ПРАНДТЛЯ

Аннотация—Рассмотрены три различные гипотезы о турбулентном переносе: модель турбулентности ван-Дриста, модифицированная гипотеза Ни-Коважного и модель, построенная на гипотезе об энергии турбулентности и длине смешения. Разрабоана модель, учитывающая изменение турбулентного числа Прандтля. Приведены численные решения, полученные модифицированным методом конечных разностей Патанкара-Сполдинга. Рассчитаны некоторые эмпирические константы, которые затем сравниваются с экспериментальными данными, причем сравнение показало хорошее соответствие расчета и эксперимента. Показана важность учета изменения турбулентного числа Прандтля.